10. 设$(M,\mathscr{D})$是n维C^∞流形，$\mathscr{D}=\{(U,\varphi)\}$。$T(M)=\bigcup_{p\in M}T_p(M)=\{X_p \mid X_p\in T_p(M), p\in M\}$。设投影

$$\pi_1: T(M)\to M, \pi_1(X_p)=p.$$

如果(U,φ)，$\{x^i\}$是M上的局部坐标系，$U^*=\pi_1^{-1}(U)$。命

$$\pi_2: U^*\to R^n, \pi_2(X_p)=(a^1,\cdots,a^n)\left(\text{这里 } X_p=\sum_{i=1}^n a^i\frac{\partial}{\partial x^i}\right).$$

φ^*: $U^*\to R^{2n}$, $\varphi^*(X_p)=(\varphi\circ\pi_1(X_p),\pi_2(X_p))=(\varphi(p),\pi_2(X_p))$.

(1°) 证明 $\{(U^*,\varphi^*)\}$确定了$T(M)$上的一个C^∞微分构造$\mathscr{D}^*$，我们称$(T(M),\mathscr{D}^*)$为M上的切丛(它是$2n$维的C^∞流形)。

(2°) 证明 M上的C^∞向量场X是$M\to T(M)$的C^∞映射。

(3°) 如果 F: $M_1\to M_2$是C^∞映射，则F_*是$T(M_1)\to T(M_2)$的C^∞映射。

(4°) 如果$(M,\mathscr{D})=(R^n,\mathscr{D}_0)$是通常的$C^\infty$流形，证明$(T(R^n),\mathscr{D}_0^*)$就是$R^n\times R^n=R^{2n}$上的通常的$C^\infty$流形。

(1) (U^*,φ^*) , (V^*,ψ^*)

$\psi^*\circ(\varphi^*)^{-1}: \varphi^*(U^*\cap V^*)\to\psi^*(U^*\cap V^*)$

$(x^1\cdots x^n, a^1\cdots a^n)\mapsto(\varphi^{-1}(x^1,\cdots x^n), X_p)$

$\mapsto(y^1,\cdots,y^n, \sum_{i=1}^n a^i\frac{\partial y^1}{\partial x^i}, \cdots, \sum_{i=1}^n a^i\frac{\partial y^n}{\partial x^i})$

$(y^1\cdots y^n)=\psi\circ\varphi^{-1}(x^1\cdots x^n)$

$X_p=(\varphi^{-1})_*(a^i\frac{\partial}{\partial x^i})$

$\psi_*(X_p)=(\psi\circ\varphi^{-1})_*(a^i\frac{\partial}{\partial x^i})$

$=a^i\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$

(2) M的chart (U,φ)，TM的chart (V^*,ψ^*)，X在U上表示为 $\sum a^i\frac{\partial}{\partial x^i}$

$\psi^*\circ X\circ\varphi^{-1}: (x^1,\cdots x^n)\mapsto p\mapsto X_p\mapsto(x^1,\cdots x^n, a^1,\cdots,a^n)$, smooth

且$\forall X_p$的局部坐标(V^*,ψ^*)，$\exists p$的局部坐标(V,ψ)，$X_V\subset V^*$，

故$X: M\to TM$光滑

(3) $F_*X=Y$，$\forall Y_{F(p)}$的局部坐标(V^*,ψ^*)，$\exists p$的局部坐标(U,φ)，

$F(U)\subset V$，进而 $F_*(U^*)\subset V^*$

对于 TM_2中的(V^*,ψ^*)，TM_1中的(U^*,φ^*)，

$\psi^*\circ F_*\circ(\varphi^*)^{-1}: (x^1,\cdots x^n, a^1\cdots a^n)\mapsto(p, X_p)\mapsto(F(p), F_*X_{F(p)})$

$\mapsto (y^1, \cdots y^n, a^i \frac{\partial y^1}{\partial x^i}, \cdots \quad a^i \frac{\partial y^n}{\partial x^i})$ Smooth

$(y^1 \cdots y^n) = \psi \circ F \circ \varphi^{-1}(x^1, \cdots x^n)$

指的是 $y^i \frac{\partial}{\partial x^i}$

(4) $f: T\mathbb{R}^n \to \mathbb{R}^{2n}$, $(x^1, \cdots x^n, y^1 \cdots y^n) \mapsto (x^1, \cdots, x^n, y^1 \cdots y^n)$

$g: \mathbb{R}^n \to T\mathbb{R}^n$, $(x^1, \cdots x^n, y^1 \cdots y^n) \mapsto (x^1, \cdots, x^n, y^1 \cdots y^n)$

f, g 光滑, $f \circ g = Id_{T\mathbb{R}^n}$, $g \circ f = Id_{\mathbb{R}^{2n}}$

4. 设M是R^n的k维C^∞正则子流形,

$$I: M \to R^n,$$

是包含映射。$\{u^i | i=1,\cdots,k\}$为M的局部坐标系,$\{x^i | i=1,\cdots,n\}$为R^n的通常的整体坐标。则由第二章§4.1(4)得到

$$\begin{pmatrix} I_*\left(\frac{\partial}{\partial u^1}\right) \\ \vdots \\ I_*\left(\frac{\partial}{\partial u^k}\right) \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^1} \\ \cdots & \cdots & \cdots \\ \frac{\partial x^1}{\partial u^k} & \cdots & \frac{\partial x^n}{\partial u^k} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix}$$

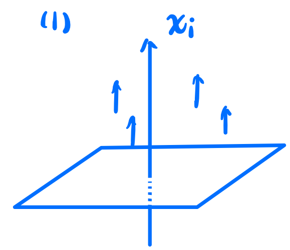
研究具体例子:

(1°) 画出R^n中整体C^∞向量场$\frac{\partial}{\partial x^i}$的示意图。

(2°) 在R^2中,取p点($p \neq (0,0)$)的局部坐标系(r,θ)(极坐标),画出局部C^∞向量场$\frac{\partial}{\partial r}$和$\frac{\partial}{\partial \theta}$的示意图(其中$(x^1,x^2)=(r\cos\theta, r\sin\theta)$)。

说明$\frac{\partial}{\partial r}$和$\frac{\partial}{\partial \theta}$是$R^2-\{(0,0)\}$上的$C^\infty$向量场的理由。并将它们表示为$\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$的线性组合。

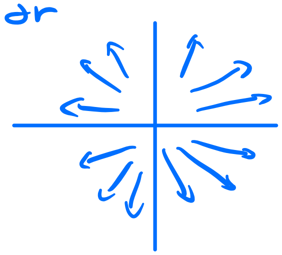
(1)



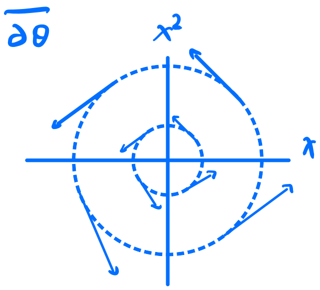
(2) $\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2}$

$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x^1} + r\cos\theta \frac{\partial}{\partial x^2}$

$\frac{\partial}{\partial r}$



$\frac{\partial}{\partial \theta}$



(3°) 在 R^3 中，取 p 点($p\neq(0,0,z)$) 的局部坐标系 (r,θ,z) (柱坐标)，画出局部 C^∞ 向量场 $\frac{\partial}{\partial r}$， $\frac{\partial}{\partial\theta}$ 和 $\frac{\partial}{\partial z}$ 的示意图(其中 $(x^1,x^2,x^3)=(r\cos\theta, r\sin\theta,z)$)。

说明这些 C^∞ 向量场可延拓到多大范围。并将它们表示为 $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ 的线性组合。

(4°) 在 R^3 中，取 p 点($p\neq(0,0,z)$)的局部坐标系 (r,θ,φ) (球坐标)，画出局部 C^∞ 向量场 $\frac{\partial}{\partial r}$， $\frac{\partial}{\partial\theta}$ 和 $\frac{\partial}{\partial\varphi}$ 的示意图(其中 $(x^1,x^2,x^3)=(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$)。

说明这些 C^∞ 向量场可延拓到多大范围. 并将它们表示为 $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ 的线性组合。

(5°) 证明(1°)—(4°) 中各坐标系都是正交坐标系 (即坐标向量彼此正交)。

(6°) 设单位圆 $M=S^1=\{(x^1,x^2)\mid(x^1)^2+(x^2)^2=1\}$，$\{\theta\}$ 为 S^1 的局部坐标系，这里 $(x^1,x^2)=(\cos\theta,\sin\theta)$。

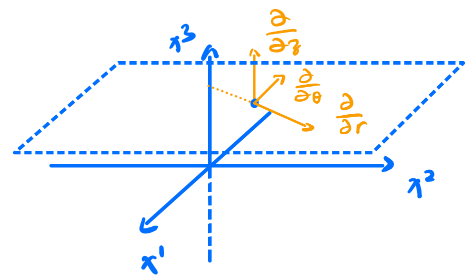
将 $I_*\left(\frac{\partial}{\partial\theta}\right)$ 表示为 $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$ 的线性组合，证明 $I_*\left(\frac{\partial}{\partial\theta}\right)$ 与 $\cos\theta\frac{\partial}{\partial x^1}+\sin\theta\frac{\partial}{\partial x^2}$ 正交. 说明 $\frac{\partial}{\partial\theta}$ 是 S^1 上的整体的 C^∞ 基向量场. 画出 $I_*\left(\frac{\partial}{\partial\theta}\right)$ 的示意图。

(7°) 设圆柱面 $M=\{(x^1,x^2,x^3)\mid(x^1)^2+(x^2)^2=1\}$，$\{\theta,z\}$ 为 M 的局部坐标系，这里 $(x^1,x^2,x^3)=(\cos\theta,\sin\theta,z)$。

(3) $\frac{\partial}{\partial r}=\cos\theta\frac{\partial}{\partial x^1}+\sin\theta\frac{\partial}{\partial x^2}$

$\frac{\partial}{\partial\theta}=-r\sin\theta\frac{\partial}{\partial x^1}+r\cos\theta\frac{\partial}{\partial x^2}$

$\frac{\partial}{\partial z}=\frac{\partial}{\partial x^3}$

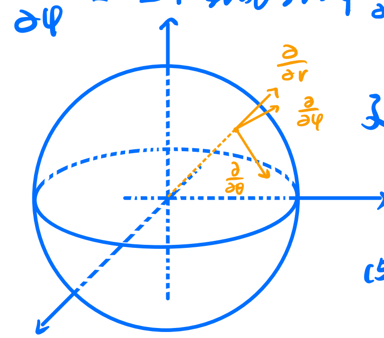


延拓到 $R^3\setminus\{(0,0,x^3)\mid x^3\in\mathbb{R}\}$

(4) $\frac{\partial}{\partial r}=\sin\theta\cos\varphi\frac{\partial}{\partial x^1}+\sin\theta\sin\varphi\frac{\partial}{\partial x^2}+\cos\theta\frac{\partial}{\partial x^3}$

$\frac{\partial}{\partial\theta}=r\cos\theta\cos\varphi\frac{\partial}{\partial x^1}-r\cos\theta\sin\varphi\frac{\partial}{\partial x^2}-r\sin\theta\frac{\partial}{\partial x^3}$

$\frac{\partial}{\partial\varphi}=-r\sin\theta\sin\varphi\frac{\partial}{\partial x^1}+r\sin\theta\cos\varphi\frac{\partial}{\partial x^1}$



延拓至 $\{(0,0,z)\}^c$

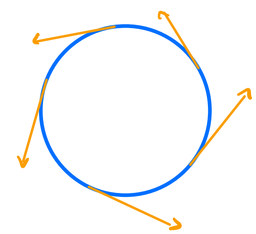
(5) 略

(8°) 设单位球面 $M=S^2=\{(x^1,x^2,x^3)\mid(x^1)^2+(x^2)^2+(x^3)^2=1\}$，$\{\theta,\varphi\}$ 为 S^2 的局部坐标系，这里 $(x^1,x^2,x^3)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$。将 $I_*\left(\frac{\partial}{\partial\theta}\right)$ 和 $I_*\left(\frac{\partial}{\partial\varphi}\right)$ 表示为 $\left\{\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right\}$ 的线性组合，证明 $I_*\left(\frac{\partial}{\partial\theta}\right)$，$I_*\left(\frac{\partial}{\partial\varphi}\right)$ 与 $\sin\theta\cos\varphi\frac{\partial}{\partial x^1}+\sin\theta\sin\varphi\frac{\partial}{\partial x^2}+\cos\theta\frac{\partial}{\partial x^3}$ 彼此正交，说明 C^∞ 向量场 $\frac{\partial}{\partial\theta}$ 和 $\frac{\partial}{\partial\varphi}$ 可延拓到多大范围。$\left\{\frac{\partial}{\partial\theta},\frac{\partial}{\partial\varphi}\right\}$ 是 S^2 上的整体的 C^∞ 基向量场吗？画出 $I_*\left(\frac{\partial}{\partial\theta}\right)$ 和 $I_*\left(\frac{\partial}{\partial\varphi}\right)$ 的示意图。

(6) $I_*\left(\frac{\partial}{\partial\theta}\right) = -\sin\theta\frac{\partial}{\partial x^1}+\cos\theta\frac{\partial}{\partial x^2}$

正交显然

$\forall\theta_0$，$\left.\frac{\partial}{\partial\theta}\right|_{\theta_0}\neq 0$，故是基向量场



(7) $I_*\left(\frac{\partial}{\partial\theta}\right) = -\sin\theta\frac{\partial}{\partial x^1}+\cos\theta\frac{\partial}{\partial x^2}$，

$I_*\left(\frac{\partial}{\partial z}\right)=\frac{\partial}{\partial x^3}$，正交显然

在 M 上 $\forall p$ 处，$\frac{\partial}{\partial\theta}$，$\frac{\partial}{\partial z}$ 线性无关，是 T_pM 的基

(8) $I_*\left(\frac{\partial}{\partial\theta}\right) = \cos\theta\cos\varphi\frac{\partial}{\partial x^1}-\cos\theta\sin\varphi\frac{\partial}{\partial x^2}-\sin\theta\frac{\partial}{\partial x^3}$

$I_*\left(\frac{\partial}{\partial\varphi}\right) = -\sin\theta\sin\varphi\frac{\partial}{\partial x^1}+\sin\theta\cos\varphi\frac{\partial}{\partial x^2}$，$\theta=\pi$，$\frac{\partial}{\partial\varphi}=0$，不是整体基向量